Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(00)00077-0
J. Appl. Maths Mechs, Vol. 64, No. 4, pp. 521–526, 2000

0021-8928/00/$—see front matter

THE STABILITY OF A CLASS OF NON-LINEAR SYSTEMS†

A. Yu. ALEKSANDROV

St Petersburg

e-mail: alex@vrm.apmath spbu.ru

(*Received* 15 *December* 1998)

The stability of the solutions of a certain class of non-linear systems is considered in the first approximation. The method of Lyapunov functions is used to obtain the sufficient conditions for stability with respect to all the variables and asymptotic stability with respect to part of the variables in the case when the first-approximation system is essentially non-linear. © 2000 Elsevier Science Ltd. All rights reserved.

1. Consider the system of differential equations

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}) + \mathbf{G}(t, \mathbf{X}, \mathbf{Y})$$
$$\dot{\mathbf{Y}} = \mathbf{Q}(t, \mathbf{Y}) + \mathbf{D}(t, \mathbf{X}, \mathbf{Y}) \tag{1.1}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are vectors of dimensions n and k, respectively and the components of the vector $\mathbf{F}(\mathbf{X})$ are continuous functions, homogeneous of order μ, $\mu \geqslant 1$; the vector functions $\mathbf{Q}(t, \mathbf{Y})$, $\mathbf{G}(t, \mathbf{X}, \mathbf{Y})$, $\mathbf{D}(t, \mathbf{X}, \mathbf{Y})$ are defined and continuous for

$$t \geqslant 0, \quad \|\mathbf{X}\| \leqslant h, \quad \|\mathbf{Y}\| \leqslant h \tag{1.2}$$

(where h is a positive constant) and satisfy the conditions

$$\mathbf{Q}(t, \mathbf{0}) \equiv \mathbf{0}, \quad \|\mathbf{G}\| \leqslant c_1(\mathbf{X}, \mathbf{Y})\|\mathbf{X}\|^{\mu}, \quad \|\mathbf{D}\| \leqslant c_2\|\mathbf{X}\|^{\lambda}$$

where $c_1(\mathbf{X}, \mathbf{Y}) \to 0$ as $\|\mathbf{X}\| + \|\mathbf{Y}\| \to 0$, $c_2 > 0$, $\lambda > 0$.

Along with Eqs (1.1), consider the system

$$\dot{\mathbf{X}} = \mathbf{F}(\mathbf{X}), \quad \dot{\mathbf{Y}} = \mathbf{Q}(t, \mathbf{Y}) \tag{1.3}$$

which we call the first-approximation system for (1.1).

We wish to find the conditions under which stability of the trivial solution of the first-approximation system implies that the trivial solution of system (1.1) will also be stable.

Let $\mu = 1$, $\mathbf{F}(\mathbf{X}) = \mathbf{AX}$, $\mathbf{Q}(t, \mathbf{Y}) = \mathbf{CY}$, where $\mathbf{A}$ and $\mathbf{C}$ are constant matrices. If $\mathbf{C} = \mathbf{0}$, we have the well-known Lyapunov–Malkin stability theorem in the critical case of multiple zero roots [1, pp. 108–113]. This theorem has been extended [2–4] to the case in which $\mathbf{C} \neq \mathbf{0}$, and it has been shown that if the trivial solution of the first-approximation system is stable and asymptotically $\mathbf{X}$-stable, then the same holds for the trivial solution of system (1.1) if $\lambda \geqslant 1$.

In what follows we will find the sufficient conditions for system (1.1) to the stable in the non-linear approximation.

2. Let the components of the vector $\mathbf{F}(\mathbf{X})$ be continuously differentiable functions, homogeneous of order μ, $\mu > 1$. Let us also assume that the trivial solution of system (1.3) is stable and asymptotically $\mathbf{X}$-stable, and that a Lyapunov function $V_1(t, \mathbf{Y})$ exists possessing the following properties: (1) $V_1(t, \mathbf{Y})$ is continuously differentiable for $t \geqslant 0$, $\|\mathbf{Y}\| \leqslant h$ and has bounded partial derivatives $\partial V_1/\partial y_i$, $i = 1, \ldots, k$; (2) $V_1(t, \mathbf{Y})$ is positive definite; (3) $dV_1(t, \mathbf{Y})/dt|_{(1.3)} \leqslant 0$.

†*Prikl. Mat. Mekh.* Vol. 64, No. 4, pp. 545–550, 2000.

Note that if $V_1(t, \mathbf{Y})$ has these properties, the trivial solution of the first-approximation system is uniformly stable.

Theorem 1. If

$$\lambda > \mu - 1 \tag{2.1}$$

then the trivial solution of system (1.1) is uniformly stable and asymptotically **X**-stable.

Proof. Since the trivial solution of system (1.3) is asymptotically **X**-stable, functions $V(\mathbf{X})$ and $W(\mathbf{X})$, positive definite and homogeneous of orders m and $m + \mu - 1$, respectively, exist such that

$$(\partial V / \partial \mathbf{X})^T \mathbf{F}(\mathbf{X}) = -W(\mathbf{X})$$

and moreover the function $V(\mathbf{X})$ is continuously differentiable [5, pp. 115–123, 6].

Evaluating the derivatives of $V(\mathbf{X})$ and $V_1(t, \mathbf{Y})$ along trajectories of system (1.1), we obtain the following estimates for all t, $\mathbf{X}$, $\mathbf{Y}$, in domain (1.2)

$$a_1 \|\mathbf{X}\|^m \leqslant V(\mathbf{X}) \leqslant a_2 \|\mathbf{X}\|^m$$

$$dV/dt\,|_{(1.1)} \leqslant \|\mathbf{X}\|^{m+\mu-1}(-a_3 + a_4 c_1(\mathbf{X}, \mathbf{Y})),$$

$$a_i > 0, \quad i = 1, \ldots 5$$

Consequently, a number $\gamma > 0$ exists such that, if for $t \in [t_0, t_1]$ the solution $(\mathbf{X}(t). \mathbf{Y}(t))^T$ of system (1.1) remains in the domain $\|\mathbf{X}\| < \gamma$, $\|\mathbf{Y}\| < \gamma$, then the following conditions hold in that interval

$$dV(\mathbf{X}(t))/dt \leqslant -a_3 \|\mathbf{X}(t)\|^{m+\mu-1}/2$$

$$V_1(t, \mathbf{Y}(t)) \leqslant V_1(t_0, \mathbf{Y}(t_0)) + a_5 \int_{t_0}^{t} \|\mathbf{X}(\tau)\|^\lambda \, d\tau$$

Applying the method of estimates [5], we see that then the following inequalities hold for all $t \in [t_0, t_1]$

$$\begin{aligned} &\|\mathbf{X}(t)\| \leqslant b_1 \|\mathbf{X}(t_0)\| (1 + b_2 \|\mathbf{X}(t_0)\|^{\mu-1}(t - t_0))^{-1/(\mu-1)} \\ &V_1(t, \mathbf{Y}(t)) \leqslant V_1(t_0, \mathbf{Y}(t_0)) + b_3 \|\mathbf{X}(t_0)\|^{\lambda-\mu+1} \end{aligned} \tag{2.2}$$

where the positive constants b_1, b_2 and b_3 are independent of the initial data of the solution.

Suppose we are given an arbitrarily small number ε, $0 < \varepsilon < \gamma$. Let

$$\beta = \inf_{t \geqslant 0, \|\mathbf{Y}\| = \varepsilon} V_1(t, \mathbf{Y})$$

Choose $\delta > 0$ such that the following conditions hold

$$2b_3 \delta^{\lambda-\mu+1} < \beta, \quad b_1 \delta < \varepsilon, \quad V_1(t, \mathbf{Y}) < \beta/2 \quad \text{for} \quad \|\mathbf{Y}\| < \delta$$

It follows from estimates (2.2) that if the initial data of the solution $(\mathbf{X}(t), \mathbf{Y}(t))^T$ satisfy the inequalities $t_0 \geqslant 0$, $\|\mathbf{X}(t_0)\| < \delta$, $\|\mathbf{Y}(t_0)\| < \delta$, then for all $t \geqslant t_0$ we have $\|\mathbf{X}(t)\| < \varepsilon$, $\|\mathbf{Y}(t)\| < \varepsilon$, and at the same time $\|\mathbf{X}(t)\| \to 0$ as $t \to +\infty$. The theorem is proved.

Remark. Theorem 1 states that if $\mu > 1$, the trivial solution will be stable for $\lambda < \mu$ also. Previously [2–4] it was assumed, for Eqs (1.1) with a linear stationary first-approximation system, that $\lambda \geqslant 1$, that is, $\lambda \geqslant \mu$. It can be shown that the propositions established in [2–4] also hold for $\lambda > 0$. Thus, inequality (2.1) is also a sufficient condition for uniform stability and asymptotic **X**-stability in the case when $\mu = 1$.

Let us assume now that the vector function $\mathbf{F}(\mathbf{X})$ in system (1.1) is twice continuously differentiable and that the vector $\mathbf{G}(t, \mathbf{X}, \mathbf{Y})$ may be expressed as

$$\mathbf{G}(t, \mathbf{X}, \mathbf{Y}) = \mathbf{B}(t)\mathbf{R}(\mathbf{X}) + \mathbf{H}(t, \mathbf{X}, \mathbf{Y})$$

where the elements of the l-dimensional vector $\mathbf{R}(\mathbf{X})$ are continuously differentiable functions, homogeneous of order σ, $\sigma \geqslant 1$ and $\mathbf{B}(t)$ is an $n \times l$ matrix which is continuous and bounded for $t \geqslant 0$ together with the integral

$$\mathbf{I}(t) = \int_0^t \mathbf{B}(\tau)d\tau \tag{2.3}$$

The vector function $\mathbf{H}(t, \mathbf{X}, \mathbf{Y})$ is assumed to be continuous in the domain (1.2) and to satisfy the inequality $\|\mathbf{H}\| \leqslant c_3(\mathbf{X}, \mathbf{Y}) \|\mathbf{X}\|^\mu$, where $c_3(\mathbf{X}, \mathbf{Y}) \to 0$ as $\|\mathbf{X}\| + \|\mathbf{Y}\| \to 0$.

Applying Theorem 1, we find the sufficient conditions for uniform stability and asymptotic $\mathbf{X}$-stability of the trivial solution: $\sigma > \mu$, $\lambda > \mu - 1$.

The restriction thus obtained for the parameter σ may be weakened by using a construction, proposed in [7, 8], of non-stationary Lyapunov functions for non-linear systems.

Theorem 2. If the inequality $2\sigma > \mu + 1$, $\lambda > \mu - 1$ holds, the trivial solution of system (1.1) is uniformly stable and asymptotically $\mathbf{X}$-stable.

Proof. Let $V(\mathbf{X})$ and $V_1(t, \mathbf{Y})$ be Lyapunov functions for the first-approximation system. Since $\mathbf{F}(\mathbf{X})$ is twice continuously differentiable, we may assume that $V(\mathbf{X})$ is also twice continuously differentiable [5, pp. 119–123: 6].

Choose a Lyapunov function for system (1.1) in the form

$$V_2(t, \mathbf{X}) = V(\mathbf{X}) - (\partial V / \partial \mathbf{X})^T \mathbf{I}(t)\mathbf{R}(\mathbf{X})$$

For all t, $\mathbf{X}$ and $\mathbf{Y}$ in domain (1.2), the following inequalities hold

$$a_1 \|\mathbf{X}\|^m - a_3 \|\mathbf{X}\|^{m+\sigma-1} \leqslant V_2(t, \mathbf{X}) \leqslant a_2 \|\mathbf{X}\|^m + a_3 \|\mathbf{X}\|^{m+\sigma-1}$$

$$dV_2 / dt|_{(1.1)} \leqslant \|\mathbf{X}\|^{m+\mu-1} (-a_4 + a_5 \|\mathbf{X}\|^{\sigma-1} + a_6 \|\mathbf{X}\|^{2\sigma-\mu-1} + a_7 c_3(\mathbf{X}, \mathbf{Y}))$$

$$a_i > 0, \quad i = 1, \dots, 7$$

Using the functions $V_1(t, \mathbf{Y})$ and $V_2(t, \mathbf{X})$, we continue the proof as in the case of Theorem 1.

Thus, the trivial solution may also be stable for $\sigma \leqslant \mu$.

Now consider the system

$$\begin{aligned} \dot{\mathbf{X}} &= \partial W / \partial \mathbf{X} + \mathbf{S}(\mathbf{Y})\mathbf{X} + \mathbf{B}(t)\mathbf{R}(\mathbf{X}) \\ \dot{\mathbf{Y}} &= \mathbf{Q}(t, \mathbf{Y}) + \mathbf{D}(t, \mathbf{X}, \mathbf{Y}) \end{aligned} \tag{2.4}$$

where $W(\mathbf{X})$ is a continuously differentiable negative definite function, homogeneous of order $\mu + 1$, $\mu > 1$ and $\mathbf{S}(\mathbf{Y})$ is a skew-symmetric matrix, continuous for $\|\mathbf{Y}\| \leqslant h$, such that $\mathbf{S}(\mathbf{0}) = \mathbf{0}$; the matrix $\mathbf{B}(t)$ and the vectors $\mathbf{R}(\mathbf{X})$, $\mathbf{Q}(t, \mathbf{Y})$, $\mathbf{D}(t, \mathbf{X}, \mathbf{Y})$ satisfy the conditions specified above. As before, we will assume that the system $\dot{\mathbf{Y}} = \mathbf{Q}(t, \mathbf{Y})$ has a Lyapunov function $V_1(t, \mathbf{Y})$ with properties 1–3.

Theorem 3. If $\sigma \geqslant \mu$, $\lambda > \mu - 1$, the trivial solution of system (2.4) is uniformly stable and asymptotically $\mathbf{X}$-stable.

To prove this, we must choose the Lyapunov function in the form

$$V(t, \mathbf{X}) = \frac{1}{2}\mathbf{X}^T\mathbf{X} - \mathbf{X}^T\mathbf{I}(t)\mathbf{R}(\mathbf{X})$$

and then again use the method of estimates.

3. Let us consider some examples of the use of the theorems proved above.

Example 1. Suppose the motion of a holonomic mechanical system with $n + k$ generalized coordinates $(\mathbf{q}, \mathbf{s})^T = (q_1, \dots, q_n, s_1, \dots, s_k)^T$ is described by the equations

$$\ddot{\mathbf{q}} = -\partial P_1 / \partial \mathbf{q} + \partial W / \partial \dot{\mathbf{q}} + \mathbf{G}(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{s}, \dot{\mathbf{s}})$$
$$\ddot{\mathbf{s}} = -\partial P_2 / \partial \mathbf{s} + \mathbf{D}(t, \mathbf{q}, \dot{\mathbf{q}}, \mathbf{s}, \dot{\mathbf{s}}) \tag{3.1}$$

where $P_1(\mathbf{q})$ is a continuously differentiable positive-definite function, homogeneous of order $\nu + 1$, $\nu > 1$, $W(\dot{\mathbf{q}})$ is a continuously differentiable negative-definite function, homogeneous of order $\mu + 1$, $\mu > 1$ and $P_2(\mathbf{s})$ is a positive definite function, continuously differentiable for $\|\mathbf{s}\| \leqslant h$; the vector functions $\mathbf{G}$ and $\mathbf{D}$ are continuous in the domain $t > 0$, $\|\mathbf{q}\| \leqslant h$, $\|\dot{\mathbf{q}}\| \leqslant h$, $\|\mathbf{s}\| \leqslant h$, $\|\dot{\mathbf{s}}\| \leqslant h$ and satisfy the inequalities

$$\|\mathbf{G}\| \leqslant c_1(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{s}, \dot{\mathbf{s}}) \|\mathbf{q}\|^{\alpha}, \quad \|\mathbf{D}\| \leqslant c_2 \|\mathbf{q}\|^{\lambda}$$

where $c_1(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{s}, \dot{\mathbf{s}}) \to 0$ as $\|\mathbf{q}\| + \|\dot{\mathbf{q}}\| + \|\mathbf{s}\| + \|\dot{\mathbf{s}}\| \to 0$, and α, λ and c_2 are positive constants.

We will investigate conditions of stability of the equilibrium position

$$\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}, \quad \mathbf{s} = \dot{\mathbf{s}} = \mathbf{0} \tag{3.2}$$

Consider the function

$$V(\mathbf{q}, \dot{\mathbf{q}}) = \left(\frac{1}{2}\dot{\mathbf{q}}^T\dot{\mathbf{q}} + P_1(\mathbf{q})\right)^{\beta} + cP_1^r(\mathbf{q})\dot{\mathbf{q}}^T\mathbf{q}$$

where $c > 0$, $\beta \geqslant 1$, $r \geqslant 1$.

If

$$r - \beta + 1 = \max\left\{\frac{\mu - 1}{2}, \ \frac{\nu - \mu}{\mu(\nu + 1)}\right\} \tag{3.3}$$

then for sufficiently small c the function V will be positive-definite, and its derivative along trajectories of the system

$$\ddot{\mathbf{q}} = -\partial P_1 / \partial \mathbf{q} + \partial W / \partial \dot{\mathbf{q}} \tag{3.4}$$

will be a negative-definite function; moreover δ, $0 < \delta < 1$ exists, such that, for $\|\mathbf{q}\| < \delta$, $\|\dot{\mathbf{q}}\| < \delta$,

$$a_1(\|\mathbf{q}\|^{\beta(\nu+1)} + \|\dot{\mathbf{q}}\|^{2\beta}) \leqslant V(\mathbf{q}, \dot{\mathbf{q}}) \leqslant a_2(\|\mathbf{q}\|^{\beta(\nu+1)} + \|\dot{\mathbf{q}}\|^{2\beta})$$
$$dV/dt|_{(3.4)} \leqslant -a_3(\|\mathbf{q}\|^{(\nu+1)(r+1)} + \|\dot{\mathbf{q}}\|^{2\beta+\mu-1})$$

where a_1, a_2 and a_3 are positive constants.

Taking condition (3.3) into account, we have

$$dV/dt|_{(3.4)} \leqslant -a_3(\|\mathbf{q}\|^{(\nu+1)(r+1)} + \|\dot{\mathbf{q}}\|^{2(r+1)}) \leqslant -a_3a_4(\|\mathbf{q}\|^{\beta(\nu+1)} + \|\dot{\mathbf{q}}\|^{2\beta})^{\gamma} \leqslant -a_5V^{\gamma}$$

$$a_4 = \min_{z_1^2+z_2^2=1}(|z_1|^{2\gamma} + |z_2|^{2\gamma}), \quad a_5 = a_3a_4a_2^{-\gamma}, \quad \gamma = \frac{r+1}{\beta} > 1$$

Applying the method of estimates [5, pp. 70–72], we see that solutions of system (3.4) beginning at $t = 0$ in a sufficiently small neighbourhood of the point $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ will satisfy the following conditions for all $t \geqslant 0$

$$\|\mathbf{q}(t)\|^{(\nu+1)(r-\beta+1)} \leqslant A/(t+1), \quad \|\dot{\mathbf{q}}(t)\|^{2(r-\beta+1)} \leqslant A/(t+1), \quad A > 0$$

For the system

$$\ddot{\mathbf{s}} = -\partial P_2 / \partial \mathbf{s}$$

we choose a Lyapunov function in the form

$$V_1(\mathbf{s}, \dot{\mathbf{s}}) = \frac{1}{2}\dot{\mathbf{s}}^T\dot{\mathbf{s}} + P_2(\mathbf{s})$$

Using the functions just constructed, it can be shown that if

$$\alpha \geqslant \max\left\{\nu, \frac{\mu(\nu+1)}{2}\right\}, \quad \lambda > \max\left\{\frac{\nu}{\mu} - 1, \frac{(\mu-1)(\nu+1)}{2}\right\}$$

the equilibrium position (3.2) of system (3.1) is uniformly stable with respect to all the variables and asymptotically stable with respect to $\mathbf{q}$, $\dot{\mathbf{q}}$.

Example 2. Suppose we are given a rigid body rotating in inertial space at an angular velocity ω about its centre of inertia O. Let $Oxyz$ be coordinate axes attached to the body, coinciding with its principal central axes. The equations of rotational motion of the body under the action of a moment $\mathbf{M}$ have the form

$$\Theta\dot{\omega} + \omega \times \Theta\omega = \mathbf{M}$$

where Θ is the inertia tensor of the body, $\Theta = \mathrm{diag}\{A_1, A_2, A_3\}$ [9].

Let us consider the motion of the rigid body in a resistant medium.

We will assume that the moment of the dray forces of the medium is defined by the formula $\mathbf{M} = (\gamma_1\omega_1^\mu, \gamma_2\omega_2^\mu, 0)^T$, where γ_1 and γ_2 are negative constants and μ is a rational number with odd numerator, $\mu > 1$. We will also assume that $A_1 > A_3$ and $A_2 > A_3$.

Using the function

$$V = A_1(A_1 - A_3)\omega_1^2 + A_2(A_2 - A_3)\omega_2^2$$

it can be showing [3, pp. 19–20] that the equilibrium position $\omega = \mathbf{0}$ is asymptotically stable with respect to ω_1, ω_2.

Suppose that, besides the moment $\mathbf{M}$, the body is subject to another moment, due to perturbing forces $\mathbf{M}_1 = \mathbf{B}(t)\mathbf{R}(\omega_1, \omega_2)$, where the components of the l-dimensional vector $\mathbf{R}(\omega_1, \omega_2)$ are continuously differentiable functions, homogeneous of order σ, $\sigma \geqslant 1$, and $\mathbf{B}(t)$ is a $3 \times l$ matrix which is continuous and bounded for $t \geqslant 0$ together with the integral (2.3).

Applying Theorem 3, we see that if $\sigma \geqslant \mu$, $1 < \mu < 3$, the equilibrium position $\omega = \mathbf{0}$ of the perturbed system is uniformly stable with respect to all the variables and asymptotically stable with respect to ω_1 and ω_2.

If the last component of the vector $\mathbf{M}_1$ is zero, the Lyapunov function can be chosen in the form

$$V = \frac{1}{2}\omega^T\Theta\omega - \omega^T\mathbf{I}(t)\mathbf{R}(\omega_1, \omega_2)$$

If $\sigma \geqslant \mu$, the function we have constructed satisfies all the conditions of Rumyantsev's theorem on asymptotic stability with respect to part of the variables [10, p. 38]. Consequently, in this case the equilibrium position will be uniformly stable with respect to all the variables and asymptotically stable with respect to ω_1 and ω_2 for $\mu \geqslant 3$ too.

REFERENCES

1. MALKIN, I. G., *Theory of Stability of Motion*. Nauka, Moscow, 1952.
2. OZIRANER, A. S., The stability of motion in critical cases. *Prikl. Mat. Mekh.*, 1975, **39**, 3, 415–421.
3. VOROTNIKOV, V. I., *The Stability of Dynamical Systems with Respect to Part of the Variables*. Nauka, Moscow, 1991.
4. MATROSOVA, N. I., Lyapunov vector-functions in the study of critical cases. In *Problems of the Qualitative Theory of Differential Equations*. Nauka, Novosibirsk, 1988, pp. 195–203.
5. ZUBOV, V. I., *The Stability of Motion*. Vysshaya Shkola, Moscow, 1973.
6. KRASOVSKII, N. N., Stability according to the first approximation. *Prikl. Mat. Mekh.*, 1955, **19**, 5, 516–530.
7. ALEKSANDROV, A. Yu., The stability of the equilibrium of non-stationary systems. *Prikl. Mat. Mekh.*, 1996, **60**, 2, 205–209.
8. ALEKSANDROV, A. Yu., The asymptotic stability of the equilibrium of non-autonomous systems. *Prikl. Mat. Mekh.*, 1998, **62**, 1, 35–40.

9. LUR'VE, A. I., *Analytical Mechanics*. Fizmatgiz, Moscow, 1961.
10. RUMYANTSEV, V. V. and OZIRANER, A. S., *Stability and Stabilization of Motion with Respect to Part of the Variables*. Nauka, Moscow, 1987.

Translated by D.L.